CIRCULATION WITHIN VISCOUS DEFORMED DROPS MOVING AT A CONSTANT VELOCITY IN A GAS

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We examine the motion of a liquid drop in a steady-state flow, when the ratio of the dynamic viscosity of the external flow to the viscosity of the internal flow is small. The internal circulation is determined on the basis of an established bilateral evaluation of the energy dissipation within the drop. The relationship between the internal circulation and the deformation of the drop is established from the equation for the balance of the external and internal pressure forces and the forces of surface tension. We have derived equations for the vibrations of the drop falling in the gas, and we have solved the stability problem. The theoretical value for the critical size of the drop is in agreement with experimental data on droplet destruction.

1. Expansion Over a Small Viscosity Ratio. The first-approximation equations. Let us examine the axisymmetric streamlining of a liquid drop. The liquid inside and outside the drop is assumed to be viscous and incompressible. We will denote the velocity vector $\mathbf{v}_+$ outside of the channel and μ_+ and ν_+ will denote the dynamic and kinematic viscosities; ρ_+ represents the density of the liquid outside of the drop. The corresponding characteristics of motion within the drop will be identified with the subscript minus sign.

Inside and outside of the drop the velocity fields are subject to the Navier–Stokes equations. At the drop surface ∂V four conditions are specified: the equality to zero for the normal velocities v_n, the continuity of the tangential velocities v_τ, and for the tangential stress σ_τ; at infinity the condition $\mathbf{v}_+ \to \mathbf{v}_\infty$ is fulfilled. We assume the shape of the drop to be known. To determine the shape of the drop we resort, in Sec. 6, to the boundary condition for normal stresses.

For droplets moving in a gas we generally satisfy the condition $\mu_+/\mu_- \ll 1$ in which v_- is considerably smaller than v_∞. With a large Reynolds number Re_+ the external flow near the boundary exhibits a structure characteristic for the boundary-layer theory applicable to a solid surface. The boundary-layer thickness $\delta_+ = \ell/\sqrt{\mathrm{Re}_+}$ (ℓ is the radius of a sphere equivalent in volume to that of the droplet).

The characteristic value of $\sigma_{+\tau} \sim \sigma$ is inversely proportional to δ_+:

$$\sigma = \mu_+ v_\infty/\delta_+ = \mu_+ v_\infty \sqrt{\mathrm{Re}_+}/l,\ \mathrm{Re}_+ = l v_\infty/\nu_+. \tag{1.1}$$

The tangential stress $\sigma_{+\tau}$ generates a vortex flow inside the drop, the velocity of this flow given by $v_- \sim \sigma\ell/\mu_-$, from which, by means of (1.1), we have $v_- \sim R v_\infty$, $R = (\mu_+/\mu_-)\sqrt{\mathrm{Re}_+}$. Assuming the parameter R to be small, we will look for the solution of the boundary-value problem in the form of expansions over the powers of R:

$$\mathbf{v}_+ = v_\infty\left(\mathbf{v}_+^{(0)} + R\mathbf{v}_+^{(1)} + \ldots\right),\quad \mathbf{v}_- = v_\infty\left(R\mathbf{v}_-^{(1)} + \ldots\right),$$
$$\sigma_{+\tau} = \sigma\left(\tau^{(0)} + R\tau^{(1)} + \ldots\right). \tag{1.2}$$

The expansions for the pressures begin from terms exhibiting orders of $p_+ \sim \rho_+ v_\infty^2$, $p_- \sim \rho_- v_-^2 \sim \rho_- R^2 v_\infty^2$. Having substituted these expansions into the Navier–Stokes equations and into the boundary conditions, we will find the equations and the boundary conditions for $\mathbf{v}_\pm^{(i)}$, $p_\pm^{(i)}$ (i = 0, 1, ...). For $\mathbf{v}_+^{(0)}$, $p_+^{(0)}$ we obtain the boundary-value problem for the streamlining of a solid under a condition of adhesion (the dimensionless tangential stress $\tau^{(0)}$ at the boundary is found from its solution), and for $\mathbf{v}_-^{(1)}$ we obtain the problem of the flow of a viscous fluid within a volume V with the following condition for the dimensionless tangential stress at the closed surface of the stream ∂V [1]:

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$$(\mathbf{v}_-^{(1)}\nabla)\,\mathbf{v}_-^{(1)} = -\nabla p_-^{(1)} + \frac{1}{\mathrm{Re}_-}\Delta\mathbf{v}_-^{(1)}, \quad \nabla v_-^{(1)} = 0, \tag{1.3}$$

$$\partial V\colon\ 2e_{ij}n_i\tau_j = \tau^{(0)}, \quad v_{-n}^{(1)} = 0, \quad \mathrm{Re}_- = lv_\infty R/\nu_-$$

(e_{ij}, n_i, and τ_j are the components of the strain-rate tensor, as well as of the unit vectors normal and tangential to the surface ∂V. Here, the subscripts ij are understood to refer to summation). The potential mass forces in Eqs. (1.3) have been included in the modified pressure $p_-^{(1)}$.

2. Structure of the Solution for the Internal Boundary-Value Problem in First Approximation. The solution of the problem of steady-state drop motion, where the shape of the drop is known, is determined in the general case by three dimensionless parameters Re_+, Re_-, and R. The solution of the first-approximation boundary-value problem (1.3) depends on Re_+ and Re_- [1], with Re_+ representing the only significant parameter, because it determines the distribution of $\tau^{(0)}$ at the surface of the drop. This fact is ascertained on the basis of bilateral estimates for the energy dissipation D within the drop [2]. According to (1.2), the velocity field within the drop is proportional to R, from which we have $D = \mu_- v_\infty{}^2 R^2 \ell d(\mathrm{Re}_+, \mathrm{Re}_-)$. The dimensionless dissipation function d is expressed in terms of the strain-rate tensor $e_{ij}{}^{(1)}$ for the velocity field $v_-{}^{(1)}$

$$d = 2\int\limits_V e_{ij}^{(1)}e_{ij}^{(1)}\,dV = \int\limits_{\partial V}\tau^{(0)}v_\tau^{(1)}\,dS, \tag{2.1}$$

where the second equation represents the energy dissipation equation for the work of the surface forces, and it is valid for any steady-state flow of a viscous liquid.

The following bilateral estimates from [2] are valid for the energy dissipation:

$$d(\mathrm{Re}_+,\infty) \leqslant d(\mathrm{Re}_+,\mathrm{Re}_-) \leqslant d(\mathrm{Re}_+, 0). \tag{2.2}$$

The first follows from the Lagrange principle, according to which the functional

$$I(\mathbf{v}^{(1)}) = \int\limits_V e_{ij}^{(1)}e_{ij}^{(1)}\,dV - \int\limits_{\partial V}\tau^{(0)}v_\tau^{(1)}\,dS,$$

determined in the solenoidal velocity field satisfying boundary conditions (1.3), attains its minimum value in the solution of the Stokes equations, where the nonlinear inertial forces are negligibly small. If the upper bound of (2.2) is valid for any solution of boundary-value problem (1.3), the lower bound is valid in the assumption that the function d is monotonically dependent on R_-. The estimation functions for the rather simple regions are found analytically, from solutions of the linear problems.

For a spherical droplet (the case of a small Weber number We) the bilateral estimates are written out in explicit form. For this we will present $\tau^{(0)}$ in the form of a series over associated Legendre polynomials:

$$\tau^{(0)}(\mathrm{Re}_+,\theta) = \sum_{n=1}^{\infty} a_n P_n(\cos\theta) \tag{2.3}$$

(θ is the polar angle reckoned from the trailing point on the sphere).

For $0.05 \le \mathrm{Re}_+ \le 20$ the coefficients a_n are given in [3], where the problem of the streamlining of a sphere has been solved by the Galerkin method. With small Re_+, a_n are found from the asymptotic solution of the problem of sphere streamlining when $\mathrm{Re}_+ \ll 1$.

In [4-6] we find results from numerical calculations for the cases in which $\mathrm{Re}_+ = 50$, 150, and 500 and experimental data for the function $\tau^{(0)}$ when $\mathrm{Re}_+ = 78{,}600$, which makes it possible to calculate a_n at these Re_+.

The stream function ψ_0 of the limit solution for boundary-value problem (1.3) as $\mathrm{Re}_- \to 0$ and its corresponding energy dissipation are represented as the series

$$\psi_0 = \sum_{n=1}^{\infty}\frac{r^{n+1}-r^{n+3}}{4n+2}\,a_n\sin\theta P_n^{(1)}(\cos\theta), \tag{2.4}$$

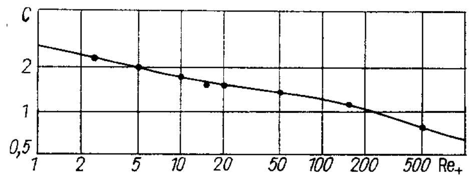

Fig. 1

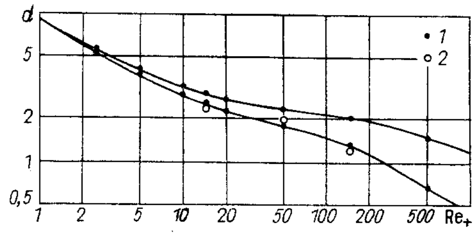

Fig. 2

$$d(\mathrm{Re}_+, 0) = 4\pi \sum_{n=1}^{\infty} \frac{n(n+1)}{(2n+1)^2} a_n^2.$$

The stream function ψ_∞ of the limit flow as $\mathrm{Re}_- \to \infty$, found in [1], coincides with the first term in series (2.4):

$$\psi_\infty = (1/6)(r^2 - r^4)a_1 \sin^2\theta,\ d(\mathrm{Re}_+, \infty) = (8/9)\pi a_1^2. \tag{2.5}$$

The limit flow (2.5) has a constant dimensionless rotation intensity within the sphere

$$C = \frac{|\mathrm{rot}\, \mathbf{v}_-^{(1)}|}{r \sin\theta} = \frac{5}{3} a_1 = \frac{5}{4} \int_0^\pi \tau^{(0)} \sin^2\theta\, d\theta. \tag{2.6}$$

This result was obtained in [1] from the second of the equations in (2.1), which may be regarded as the decisive relationship for the constant C.

Figure 1 shows the function $C(\mathrm{Re}_+)$ in logarithmic scale for Re_+ in the range $1 \le \mathrm{Re}_+ \le 1000$. The dot identifies the calculations carried out with the aid of formulas (2.6), in which the data for a_1 and $\tau^{(0)}(\mathrm{Re}_+, \theta)$ have been taken from [3-6]. For small values of $\mathrm{Re}_+ \ll 1$ the function $C(\mathrm{Re}_+)$ can be calculated by means of the asymptotic expression $C(\mathrm{Re}_+) = (5/2)(1 + (3/8)\mathrm{Re}_+)/\sqrt{\mathrm{Re}_+}$.

Figure 2 shows the upper $d(\mathrm{Re}_+, 0)$ and lower $d(\mathrm{Re}_+, \infty)$ bounds for the energy dissipation within the drop. Points 1 represent calculations on the basis of (2.4) and (2.5), carried out with the coefficients of the tangential-stress expansion [3-6], points 2 represent the values of $d(\mathrm{Re}_+, \mathrm{Re}_-)$ for $\mathrm{Re}_+ = 15$, $\mathrm{Re}_- = 16.1$; $\mathrm{Re}_+ = 50$, $\mathrm{Re}_- = 97.9$; $\mathrm{Re}_+ = 150$, $\mathrm{Re}_- = 509$, determined from the results of the numerical calculations, as taken from [4]. The deviation of these values from the asymptote $d(\mathrm{Re}_+, \infty)$ does not exceed 9% and is comparable to the error in the numerical calculations [4], obtained with a difference scheme for the Navier–Stokes equations.

When Re_- changes from zero to infinity, the function $d(\mathrm{Re}_+, \mathrm{Re}_-)$ varies within small limits which are determined by the estimates $d(\mathrm{Re}_+, 0)$ and $d(\mathrm{Re}_+, \infty)$. The change in d amounts to 40% (the highest) when $\mathrm{Re}_+ = 150$, 30% when $\mathrm{Re}_+ = 50$, about 10% when $\mathrm{Re}_+ = 10$, and less than 1% when $\mathrm{Re}_+ = 1$.

Analysis of the isolines of the stream functions (2.4) for various Re_+ shows that when $\mathrm{Re}_+ = 150$ a second vortex arises within the droplet in the vicinity of the trailing point. When $\mathrm{Re}_+ = 150$ the maximum velocity in the area of the second vortex is smaller by a factor of 30 than the greatest velocity within the droplet. Even with extremely large Re_- the Reynolds number calculated on the basis of the characteristic dimension of the second vortex and the velocity within that vortex will be small. The dividing streamline marking the boundary of the second vortex will therefore be independent of Re_-. This is confirmed by comparison with numerical calculations. The dividing streamline shown in Fig. 3a [4] for $\mathrm{Re}_+ = 150$ and $\mathrm{Re}_- = 509$ coincides in accuracy with that shown in Fig. 3b for $\mathrm{Re}_+ = 150$ and $\mathrm{Re}_- = 0$.

The attained simplification of the problem where in the place of three dimensionless parameters only one (Re_+) is significant is possible only with the following limitation: the velocity within the droplet is considerably smaller than v_∞ or $R/5 \ll 1$.

3. Equation of the Boundary Layer within the Drop. In accordance with theory [1], given a large internal Re_-, a "soft" boundary layer of thickness $\delta_- = l/\sqrt{\mathrm{Re}_-}$ is formed

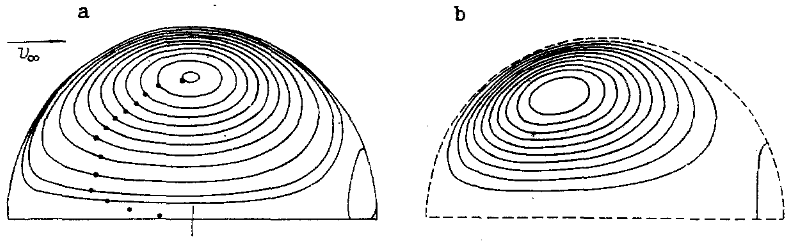

Fig. 3

around the edges of the drop. Within this boundary layer we have a change in the values of $\omega_- = |\text{rot}\, v_-|$ on the order of unity and a small change in v_- on the order of δ_-. The quantity $c_- = \omega/y$ (y is the distance from the axis of symmetry) is constant within the drop, everywhere outside of the boundary layer ($c_- = c = \text{const}$), and within the boundary layer it satisfies the equation of convective diffusion at the limit velocity field $v_c(|\text{rot}\, v_c| = cy)$.

In writing the equations of the boundary layer we will take into consideration that the derivative with respect to the coordinate x_1 across the boundary layer is larger by a factor of $\sqrt{\text{Re}_-}$ than the derivative with respect to the coordinate s along the streamline. Thus,

$$v_c \frac{\partial c_-}{\partial s} = \frac{\nu_-}{h_1^2} \frac{\partial^2 c_-}{\partial x_1^2} \tag{3.1}$$

(h_1 is the Lamé coefficient corresponding to the coordinate x_1). For the coordinate x_1 it is convenient to take the stream function ψ_0 of the velocity field $v_0 = v_c/c$, so that $h_1 = 1/(yv_0)$, while (3.1) is transformed to an equation of the heat-conduction type

$$\partial c_-/\partial t = \partial^2 c_-/\partial x^2,\ dt = y^2(s) v_0(s) ds,\ x = \psi_0 \sqrt{c/\nu_-}, \tag{3.2}$$

where y(s) represents the distance from the point at the edge of the drop to the axis of symmetry; $v_0(s)$ is the value of the velocity v_0 at this same point on the boundary.

The boundary streamline L is divided by the poles A and B into segments L_1 and L_2. For the first segment in contact with the outside we have specified σ_τ which we will express in terms of the magnitude of the vortex, the velocity, and the curvature K of the boundary streamline:

$$\sigma_\tau = \mu_-(yc_- - 2Kv_-). \tag{3.3}$$

From (3.3) we will represent c_- in terms of the stress $\sigma_{+\tau}$ at the edge of the drop, from which we obtain the condition imposed on L_1

$$c_-(t, 0) = (2Kv_c + \sigma_{+\tau}/\mu_-)/y,\quad 0 \leqslant t \leqslant t_0,\quad t_0 = \int_A^B y^2 v_0\, ds. \tag{3.4}$$

In the boundary layer in contact with the segment L_2 of the axis of drop rotation, c_- is carried along the streamline without change, and this leads us to the condition of equality for the functions c_- at the poles A and B

$$c_-(0, x) = c_-(t_0, x). \tag{3.5}$$

Outside of the boundary layer c_- must tend to the constant c and from this we have

$$c_- \to c \text{ as } x \to \infty. \tag{3.6}$$

The boundary-value problem (3.2)-(3.6) has only a single solution. If we predetermine $c_-(t, x)$ outside of the segment $t \in [0, t_0]$ from the periodicity condition $c_-(t + t_0, x) = c_-(t, x)$, we will wind up with the classical problem of the temperature oscillations of a semiinfinite rod, a problem that is solved by the Fourier method. In this case c is the zero Fourier harmonic for $c_-(t, 0)$, determined from (3.4):

$$c=\frac{1}{t_0}\int_0^{t_0} c_-(t, 0)\,dt. \quad (3.7)$$

It is not difficult to demonstrate that relationship (3.7), with consideration of (3.4), is equivalent to equality between the dissipation of energy at the limit velocity field $v_c = cv_0$ and the work performed by the force of the tangential stress $\sigma_{+\tau}$ at the surface of the drop and that it is written, in this case, as

$$c^2 D_0 = \int_{\partial V} \sigma_{+\tau} v_c\, dS \quad (3.8)$$

($c^2 D_0$ represents the dissipation of energy at the limit velocity field $v_c = cv_0$).

Relationship (3.8), from which we find the constants c, has been derived in [1], and it is conveniently represented in the form

$$c=\frac{2\pi}{D_0}J, \quad J=\int_0^{t_0}\frac{\sigma_{+\tau}dt}{y(t)}. \quad (3.9)$$

For an ellipsoidal drop

$$D_0=\frac{8\pi}{15}\mu_- l^5\frac{\chi^{2/3}(16-2\chi^2+\chi^4)}{(4+\chi^2)^2}, \quad t_0=\frac{4}{15}l^5\chi^{2/3} \quad (3.10)$$

(χ is the ratio of the axis perpendicular to the flow to the axis of the ellipsoid parallel to the flow). In order to solve the entire problem we have only to determine the integral J.

4. Calculating the Intensity of the Vortex within the Ellipsoidal Drop. We will assume the shape of the drop to be ellipsoidal. This is not an overly rough assumption. In a number of experimental studies the drops exhibited ellipsoidal shape until extremely high strains were attained, and this over a broad range of initial parameters [7-9]. The ellipsoidality assumption has been tested in theoretical studies devoted to calculation of buoyant bubbles and falling droplets.

To simplify the calculations for J in (3.9) we will assume that the functions beneath the integral sign depend on the single linear parameter represented by the curvature radius $a = \ell\chi^{4/3}$ at the pole of the ellipsoid. Indeed, the relationship between $\sigma_{+\tau}/y$ and t is at its maximum at the pole and in the vicinity of t = 0 makes the principal contribution to J, so that the second linear parameter of the ellipsoid is less significant and we can neglect dependence on this factor. These considerations allow us to draw a conclusion as to the form of the function under the integral sign:

$$\frac{\sigma_{+\tau}}{y}=\frac{\sigma}{a}T(t', \mathrm{Re}_*), \quad t'=\frac{t}{t_0}, \quad \mathrm{Re}_*=av_\infty/\nu_+, \quad \sigma=\mu_+(v_\infty/a)\sqrt{\mathrm{Re}_*}. \quad (4.1)$$

Having substituted (4.1) into (3.9), we obtain

$$\sqrt{\Omega}=\frac{cl^2}{v_\infty}=\frac{3R(4+\chi^2)^2 C(\mathrm{Re}_*)}{5\chi^2(16-2\chi^2+\chi^4)}, \quad C(\mathrm{Re}_*)=\frac{5}{3}\int_0^1 T(t', \mathrm{Re}_*)\,dt'. \quad (4.2)$$

The function $C(\mathrm{Re}_*)$ is independent of the drop deformation. For a spherical drop formulas (2.6) and (4.2) are identical for $C(\mathrm{Re}_*)$. Thus, the parameter Ω in (4.2) for a deformed drop is expressed in terms of the function C(Re) shown in Fig. 1, which we know in advance.

As we can see from (4.2), with a constant Re the relationship between Ω and the deformation χ is quite significant. Thus, in the change of the deformation from $\chi = 1$ to $\chi = 2$ Ω changes by a factor of almost 9.

The dimensionless parameter $(\rho_-/\rho_+)\Omega$ is the ratio of the characteristic dynamic pressures inside and outside of the drop. In the place of the dimensionless We the parameter $(\rho_-/\rho_+)\Omega$ determines the deformation of the drop. Thus, based on the deformation of the drop it is possible to make a judgement about $(\rho_-/\rho_+)\Omega$, and with the aid of (4.2) to find $C(\mathrm{Re}_*)$. The proposed method is one of the possible means of determining the internal intensity of the vortex directly through experimentation.

5. Calculation of the Vortex within the Drop on the Basis of Deformation. Let us examine the problem of determining the shape of a drop falling in a gas under the action of the force of gravity. We will make use in this case of that boundary condition for normal stresses that has not been taken into consideration up to this point. With $Re_+ \gg 1$ the normal stresses are close to the pressures p_+ and p_- corresponding to the limit flows of inviscid liquids. In this case the external flow, generally speaking, is a detached flow. We will write this boundary condition as follows:

$$p_- - p_+ = 2\varkappa H \tag{5.1}$$

(κ is the coefficient of surface tension and H represents the mean surface curvature).

It is well known from experimental data on falling raindrops that these drops have a shape similar to a compressed ellipsoid of revolution [8, 9]. Below we propose the Galerkin projection method in combination with a variation method from [10]. The pressure outside and inside the drop are presented in the form

$$p_+ = (1/2)\rho_+ v_\infty^2 c_p^+ - \rho_+ gz,\ p_- = (1/2)\rho_- v_\infty^2 \Omega c_p^- - \rho_- gz, \quad H = h/l,\ \varkappa H = \rho_+ v_\infty^2 h/\mathrm{We}, \tag{5.2}$$

where c_p^+ and c_p^- are the dimensionless pressures; ρ_+gz and ρ_-gz are the hydrostatic pressures outside and inside the drop; z is a coordinate directed vertically upward; We is the Weber number ($We = \rho_+ v_\infty^2 \ell/\kappa$).

In order to find the deformation of the ellipsoid we will take three functions, i.e., the Legendre polynomials P_0, P_1, and P_2, orthogonal at the ellipsoid. We will project the boundary condition (5.1) onto these functions. Projection onto P_0 determines the unknown constant in the pressure difference $p_- - p_+$. Projection onto P_1, i.e., the equation of balance for the force of drop resistance and the force of gravity, makes it possible to find the velocity v_∞ of the falling drop. Projection onto P_2, i.e., the equation of drop deformation, is given by

$$\frac{\rho_-}{\rho_+}\Omega\left(c_p^-, P_2\right) - \left(c_p^+, P_2\right) - \frac{4}{\mathrm{We}}(h, P_2) = 0. \tag{5.3}$$

The hydrostatic pressure in Eq. (5.3) adds nothing, since its projection onto P_2 is equal to zero. The projections in (5.3) are expressed in terms of the derivatives of the dimensionless functions of the kinetic energies outside of and within the drop, i.e., $m(\chi)$, $i(\chi)$, as well as of the surface area of the ellipsoid $s(\chi)$:

$$(c_p^-, P_2) = i'(\chi),\ (c_p^+, P_2) = m'(\chi),\ (h, P_2) = -(1/4)s'(\chi). \tag{5.4}$$

The form of m, i, and s has been found analytically in [12]. Thus, from (5.3) and (5.4) we obtain the equation which establishes the relationship between the three parameters:

$$f(\chi) = -m'(\chi) + (\rho_-/\rho_+)\Omega i'(\chi) + s'(\chi)/\mathrm{We} = 0 \tag{5.5}$$

(the primes here and below indicate derivatives with respect to χ). In the particular case ($\Omega = 0$) from (5.5) we have the function $We(\chi)$ [10, 11]. We will assume the experimentally derived deformation χ and We to be known, so that from (5.5) we determined Ω, while with the aid of (4.2) we can find $C^2(Re)$.

Table 1 shows the dimensionless C for large Re_+, calculated form (4.2) and (5.5). The data for ℓ, v_∞, and χ have been taken from [8, 9, 13] for drops falling in air under normal conditions ($\kappa = 72.8$ dyn/cm, $\rho_+ = 0.0012$ g/cm³, $\nu_+ = 0.15$ cm²/sec, $\rho_- = 1$ g/cm³, $\nu_- = 0.01$ cm²/sec). We can see that C varies about a value of 0.6, which is in agreement with the values established theoretically in [1] ($C \lesssim 0.7$ as $Re_+ \to \infty$) (see Fig. 1).

6. Stability of the Steady-State Motion of the Drops. We can undertake a study of drop deformation, where the drop is moving in the flow of an ideal incompressible liquid, with the aid of the Lagrange equations. An expression is given in [10] for the Lagrange functions $L(u, \chi, \dot{\chi})$ for the dynamic model of an ellipsoidal drop exhibiting two degrees of freedom: the coordinate determining the translational motion x_0, $\dot{x}_0 = u$; and the coordinate χ which determines the deformation of the drop.

The equation for the deformation of the drop is given by

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\chi}} - \frac{\partial L}{\partial \chi} = 0. \tag{6.1}$$

TABLE 1

l, mm	v_∞, cm/sec	χ	Re_+	We	c	$\partial f/\partial\chi$	$\partial f_1/\partial\chi$	Source
0,7	517	1,060	241	0,308	0,611	—	—	[13]
1,0	649	1,104	433	0,694	0,595	—	—	«
1,5	806	1,185	806	1,62	0,553	0,391	1,31	«
2,0	874	1,279	1180	2,31	0,529	—0,327	0,533	«
2,5	909	1,389	1515	3,44	0,565	—0,433	0,353	«
2,9	917	1,492	1770	4,06	0,609	—0,502	0,241	«
3,0	918	1,53	1800	4,12	0,636	—0,488	0,237	[8]
3,5	918	1,61	2144	4,82	0,67	—0,603	0,088	«
4,0	919	1,72	2450	5,51	0,75	—0,706	—0,057	«

This equation is enhanced in [10] by the law of the conservation of momentum: $\partial L/\partial u = \text{const}$. In view of the fact that for liquid drops moving in a gas the density ρ_- exceeds the density ρ_+ of the gas by an order of three, it follows from this law that the velocity of the drop is constant. Equation (5.5) has been derived in [10] for the deformation of the drop in steady-state motion, while for the stability of the steady-state motion we have the condition

$$\partial f/\partial\chi > 0. \tag{6.2}$$

We can see from the corresponding values in Table 1 that from condition (6.2) the water droplets falling in air lose stability when $\ell \gtrsim 1.88$ mm, i.e., considerably earlier than in the experiments [8, 13]. The cause of the diversion from experiment lies in the fact that with a change in the deformation of the drop the viscous resistance and the velocity of the falling droplet also change.

In order to take this effect into consideration we will assume that the resistance law with the drag factor $c_x = 0.365\chi^{1.8}$ is quadratic with respect to velocity. Hence we have the relationship between the velocity of the falling drop and deformation:

$$u = u(\chi) = \left(\frac{8}{3c_x} gl \frac{\rho_-}{\rho_+}\right)^{1/2} = 2.7\chi^{-0.9}\left(gl\frac{\rho_-}{\rho_+}\right)^{1/2}. \tag{6.3}$$

The empirical relationship (6.3) is in good agreement with the experimental data [8, 13] (see Table 1).

The following condition of stability follows from the system of equations (6.1), (6.3):

$$\partial f_1/\partial\chi > 0, \tag{6.4}$$

where

$$f_1 = -m'(\chi)(u^2(\chi))/(u^2(\chi_0)) + (\rho_-/\rho_+)\Omega i'(\chi) + s'(\chi)/\text{We}.$$

Based on criterion (6.4) the loss of stability occurs at drop deformations larger than on the basis of (6.2). This is explained by the fact that under the condition of constancy for the drop velocity the increase in drop deformation leads to an increase in the external dynamic pressure at the equator. Indeed, with an increase in drop deformation the velocity at which falls is reduced in accordance with (6.3) and the dynamic pressure will be lower than in the case of a constant velocity of fall.

The deformation of falling raindrops has been studied all the way to $\ell = 4$ mm in the experiments described in [13], whose results are presented in Table 1 ($\ell \le 2.9$ mm) and 2 ($\ell \ge 3$ mm). As we can see from Table 1, condition (6.4) in conjunction with (6.3) is satisfied for drops of all sizes with the exception of $\ell = 4$ mm. It has been established experimentally [13] that even drops with dimensions of $\ell = 4$ mm are unstable and break up into smaller droplets.

Thus, taking into consideration the relationship between the velocity of the drop and its deformation is of considerable importance in describing the vibrations of the drops and in studying the stability of steady drop shape.

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